Minimum Velocity Increment Solution for Two-Impulse Coplanar Orbital Transfer

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The most general statement of the planar orbital transfer problem defines a trajectory with arbitrarily specified end points located on elliptical (or other conic) orbits with noncoincident apsidal axes. This report presents complete and explicit optimum solutions of the two-impulse orbital transfer problem based on a minimum total velocity increment criterion. By use of hodograph (velocity) parameters, the total velocity increment for transfer is expressed as a function of one independent variable (i.e., one of the transfer orbit hodograph parameters) and the trajectory end-point conditions. In addition to the formulation of an eighth-order (octic) polynomial equation providing interior minima, the absolute total velocity increment minimum is determined by comparing the velocity increments at the end points of the variable parameter range with those obtained from the octic. As a special case, complete analytic solutions and attendant transfer characteristics are presented graphically for transfer between any specified trajectory end points lying on circular orbits.

Nomenclature

A, B, D, E, F, G, H, J = terms defining the polynomial coefficients K and K as functions of the end-point condition
a = coefficient of the velocity increment equation
b = coefficient of the velocity increment equation
C = velocity parameter of the orbit, defined by Eq. (1)
c = coefficient of the velocity increment equation
E = total potential and kinetic energy of the orbital body
e = eccentricity of the ellipse
K_i = coefficients of the general octic equation, where $i = 0, 1, 2, \ldots 8$
m = mass of the orbital body
R = velocity parameter of the orbit, defined by Eq. (2)
r = with subscript 1 or 2, radial distance between the gravitational centers of the orbital and celestial bodies; with all other subscripts, nondimensional ratio
r_0 = normalized circular orbit velocity at the final transfer point ($= v_{c2}/v_{c1}$)
T = orbital period
Δt = transfer time interval
u = normalized initial velocity increment ($= \Delta v_1/v_{c1}$)
v = linear velocity
Δv = magnitude of the vector difference between two linear velocities
w = normalized final velocity increment ($= \Delta v_2/v_{c1}$)
x = velocity parameter of the transfer orbit, normalized with respect to the circular orbit velocity at the initial transfer point ($= C_T/v_{c1}$)
x' = velocity parameter of the transfer orbit, normalized with respect to the circular orbit velocity at the final transfer point ($= C_T/v_{c2}$)
y = normalized velocity parameter of the transfer orbit ($= R_T/v_{c1}$)
z = linear velocity, normalized with respect to the circular orbit velocity at the initial transfer point (i.e., $z_{T1} = v_{T1}/v_{c1}$)
α = angular difference between the apsidal lines of the initial orbit and the transfer orbit at perigee, with sense established by that order
η = angular difference between the apsidal lines of the transfer orbit and the final orbit at perigee, with sense established by that order
K_i = coefficients of the reduced octic equation, where $i = 0, 1, 2, \ldots 8$
μ = gravitational constant for the given celestial body
σ = angular difference between the transfer orbit phases for the two transfer points
τ = normalized transfer time
ϕ = true anomaly of the body in orbit
Ψ = apsidal line of the orbit at perigee

Subscripts

A = initial orbit
B = final orbit
c = circular orbit
LL = lower limit
r = radial
T = transfer orbit
UL = upper limit
1 = point of transfer from the initial orbit to the transfer orbit
2 = point of transfer from the transfer orbit to the final orbit
ϕ = normal

I. Introduction

ORBITAL transfer of space vehicles constitutes one of the most basic problems in astronautics. In particular, for chemical propulsion systems, those transfer conditions that require the least (or minimum) total velocity change, and consequently minimum fuel mass,[1] are of special interest.

The potential field of the attracting center in the two-body transfer problem can be considered due to a regular spherical body of uniform mass distribution or due to a nonregular body of nonuniform mass distribution. In the latter case, the orbital figure in inertial space can be closely approximated by a Keplerian orbit lying within an orbital plane, with apsidal line and line-of-nodes rotating as a result of the harmonics of the potential field. In this paper, the orbital transfer problem is considered only between coplanar orbits, neglecting the apsidal line and nodal line rotations due to the potential field harmonics.

For the problem so defined, it has been shown previously[2] that orbital transfer will be achieved with minimum total velocity change by the use of one or more instantaneous

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Table 1 Analytic relations for point-to-point transfer

$$v_{T1}^2 = (v_{c1}^2/C_T)^2 + [(v_{c1}^2/C_T) - C_T]^2 \tan^2\varphi_{T1} \tag{1A}$$

$$\Delta v_T = |\Delta v_1| + |\Delta v_2| \tag{1B}$$

$$\Delta v_1^2 = (1/C_A^2 C_T^2 \sin^2\sigma)[a_1(C_A - C_T)^2 + (b_1 C_T^2 + c_1 C_T + d_1)^2] \tag{1C}$$

$$\Delta v_2^2 = (1/C_B^2 C_T^2 \sin^2\sigma)[a_2(C_B - C_T)^2 + (b_2 C_T^2 + c_2 C_T + d_2)^2] \tag{1D}$$

where

$$\sigma = (\varphi_B - \varphi_A) + (\Psi_B - \Psi_A) = (\varphi_B - \varphi_A) + (\alpha + \eta)$$

$$a_1 = v_{c1}^4 \sin^2\sigma \qquad a_2 = v_{c2}^4 \sin^2\sigma$$

$$b_1 = C_A(1 - \cos\sigma) \qquad b_2 = C_B(\cos\sigma - 1)$$

$$c_1 = (C_A^2 - v_{c1}^2)\tan\varphi_A \sin\sigma \qquad c_2 = (C_B^2 - v_{c2}^2)\tan\varphi_B \sin\sigma$$

$$d_1 = C_A(v_{c1}^2 \cos\sigma - v_{c2}^2) \qquad d_2 = C_B(v_{c1}^2 - v_{c2}^2 \cos\sigma)$$

and

$$v_{c1}^2 = (Cv_\varphi)_1 = \mu/r_1 \tag{1E}$$

$$v_{c2}^2 = (Cv_\varphi)_2 = \mu/r_2 \tag{1F}$$

$$\varphi_{T1} = \varphi_A - \alpha = \operatorname{arc\,tan}\left[\frac{1}{\sin\sigma}\left(\cos\sigma - \frac{C_T^2 - v_{c2}^2}{C_T^2 - v_{c1}^2}\right)\right] \tag{1G}$$

$$\varphi_{T2} = \varphi_B + \eta = \operatorname{arc\,tan}\left[\frac{1}{\sin\sigma}\left(\frac{C_T^2 - v_{c1}^2}{C_T^2 - v_{c2}^2} - \cos\sigma\right)\right] \tag{1H}$$

$$C_T R_T = \frac{v_{c1}^2 - C_T^2}{\cos\varphi_{T1}} = \frac{v_{c2}^2 - C_T^2}{\cos\varphi_{T2}} \tag{1I}$$

velocity increments rather than a velocity change continuous in time. Actually, the velocity change will occur during a finite burning time; however, its approximation by instantaneous velocity increments will be valid if the thrust-to-weight ratio is relatively high. Some study[3–7] has established that minimum total velocity change for coplanar orbital transfer is provided by use of two or three instantaneous velocity increments. Consequently, the general case of two-impulse orbital transfer is studied here.

The general coplanar orbital transfer problem is defined by the following two classes: 1) the terminal (or end) points of the transfer trajectory are free, i.e., may be selected a postiori to provide a desired minimum total velocity change; and 2) the terminal points of the transfer trajectory are fixed a priori and the consequent minimum total velocity change determined.

For example, consider transfer between the elliptical orbits shown in Fig. 1a. Assuming that the end points of the complete transfer trajectory need only lie anywhere on orbits A and B, what is the transfer trajectory (and number of impulses) that requires the minimal total velocity change? To date, adequate solutions have been obtained only for special, or restricted, orbital conditions.[5–8] For example, if the space figures of orbits A and B are aligned with coincident apsidal

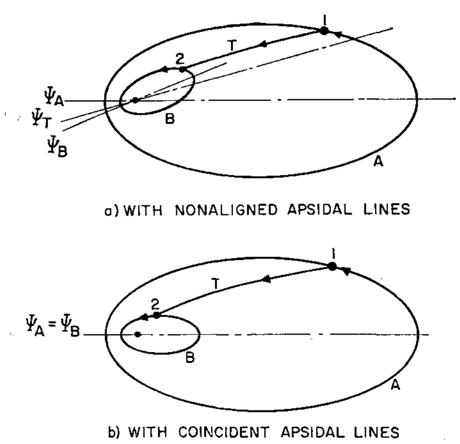

Fig. 1 Geometry of the orbital transfer problem

lines, as shown in Fig. 1b, solutions have been obtained for both the number of impulses and the location of end points that provide the desired minimum total velocity change. Such solutions are of decided interest, since they are the asymptotic limits on minimum velocity changes which may be realizable in practice.

On the other hand, if the terminal points are specified a priori, what transfer trajectory between these points will require the minimum total velocity change with two impulses? The first problem class is only one special case of the more general problem of fixed end points.

Most current work, including that of the authors, to obtain these most general solutions have been limited to two-increment transfer trajectories. Then the transfer trajectory is a segment of a transfer orbit upon which the given end points are located. Most current work, excluding that of the authors, for solutions to this two-increment transfer has been based upon Vargo's problem statement.[9] Such work[8,10] necessarily has required original and careful application of variational principles. In essence, Vargo's problem statement is comprised of 1) one equation for the total velocity change (i.e., the sum of the velocity increments at the two end points) as a function of the given end-point conditions and two apparently independent variables, subject to 2) auxiliary equations expressing the constraints imposed by the given end-point conditions.

Actually, only one variable is truly independent, because of the presence of the auxiliary conditions. To date, analyses based upon this problem statement have provided only limited success in obtaining desired general solutions.

Study of the complete transfer problem from the viewpoint of the guidance and control system designer led the authors to the use of orbital parameters, with the conic parameters transformed into hodograph parameters.[11–13] This approach enabled a problem statement that exploits orbital relations between position and velocity coordinates. In essence, this problem statement is comprised of one equation for the total velocity change as a function of the given end-point conditions and one independent variable. This independent variable is one of three parameters defining the transfer orbit. The other two parameters are dependent variables that do not enter into the total velocity change equation, by virtue of the orbital relations expressed in the modified polar hodograph.[12,13] Consequently, extremal solutions are obtainable by use of the

Table 2 Nondimensional relations for point-to-point transfer

$$z_{T1}^2 = (1/x^2) + [(1/x) - x]^2 \tan^2\phi_{T1} \tag{2A}$$

$$u^2 = \left(\frac{r_A - x}{r_A x}\right)^2 + \frac{1}{\sin^2\sigma}\left[x(1-\cos\sigma) + \left(\frac{r_A^2 - 1}{r_A}\right)\tan\phi_A \sin\sigma + \frac{(\cos\sigma - r_0^2)}{x}\right]^2 \tag{2B}$$

$$w^2 = \left(\frac{r_B - x}{r_B x}\right)^2 r_0^4 + \frac{1}{\sin^2\sigma}\left[x(\cos\sigma - 1) + \left(\frac{r_B^2 - r_0^2}{r_B}\right)\tan\phi_B \sin\sigma + \frac{(1 - r_0^2\cos\sigma)}{x}\right]^2 \tag{2C}$$

$$\phi_{T1} = \phi_A - \alpha = \arctan\left[\frac{1}{\sin\sigma}\left(\cos\sigma - \frac{x^2 - r_0^2}{x^2 - 1}\right)\right] \tag{2D}$$

$$\phi_{T2} = \phi_B + \eta = \arctan\left[\frac{1}{\sin\sigma}\left(\frac{x^2 - 1}{x^2 - r_0^2} - \cos\sigma\right)\right] \tag{2E}$$

$$xy = \frac{1 - x^2}{\cos\phi_{T1}} = \frac{r_0^2 - x^2}{\cos\phi_{T2}} \tag{2F}$$

differential calculus, without recourse to the variational calculus.

By use of the hodograph problem statement, this paper presents the following general and special solutions of the minimum velocity increment for orbital transfer: 1) the eighth-order polynomial (or octic) equation for general solutions for transfer between coplanar, nonaligned elliptical orbits; 2) the eighth-order polynomial equation for transfer solutions between circular orbits; and 3) the transfer orbit variable, transfer time, and minimum velocity increments (in graphical form) for end points on circular orbits, where the initial orbit radius is less than the final orbit radius (i.e., $r_1 < r_2$). Provision of absolute minima by the solutions of the polynomial equation is investigated and the conditions of validity established.

The analysis results enable solution for minimum total velocity increment transfer directly from given trajectory end-point conditions. Also, the graphical presentation of solutions for circular orbit transfer is discussed briefly.

II. Equation for the Total Velocity Increment

Study of the orbital transfer problem indicated that a new analytical approach would be desirable. Consequently, the use of the orbital hodograph was proposed in Ref. 11. It was shown that a planar orbit can be represented uniquely by the vehicle velocity components along the radial line between the vehicle center of mass and the celestial body center and along the normal to that radial line in the plane of the orbit.

In the most general problem for coplanar two-impulse transfer, the initial and final orbits and the initial and final transfer points are given and not selectable as convenient. The most pertinent analytical expressions of this point-to-point transfer problem are presented in Table 1 for use in the optimization. Note that the basic relations [Eqs. (1A–1F)] are functions of the independent variable C_T and the trajectory end-point conditions only. The other two parameters of the transfer trajectory (R_T, Ψ_T) are dependent upon the variable C_T, as defined by Eqs. (1G–1I). As shown previously, the following velocity parameters of the orbit are constant in magnitude:

$$C = \mu/rv_\phi \tag{1}$$

$$R = [(2E/m) + C^2]^{1/2} \tag{2}$$

The analytic relations of Table 1 are shown in nondimensional form in Table 2. Since this paper develops the minimum total velocity change for transfer, Eqs. (1B–1D) [or(1B, 2B, and 2C)] are directly useful for the optimization process.

III. Conditions for Minimum Velocity Solutions

An equation for the total velocity increment Δv_T as a function of the transfer orbit hodograph parameter C_T is available by use of Eqs. (1B–1D). The region of values for C_T is bounded, so that the functional relation must be studied at 1) interior points of relative minima, and 2) the end points of the bounded region for C_T. Note that C_T must always be a real number, since it represents a physical velocity.

The interior maxima and minima of Δv_T occur at values of C_T which are positive real roots of

$$(d/dC_T)(\Delta v_T) = 0 \tag{3}$$

Only one of these roots provides the smallest interior value of Δv_T. However, it is possible that the neighborhood of one or both end points may provide still smaller values of Δv_T than obtained from Eq. (3), as illustrated in Fig. 2. In this figure, both end-point values of Δv_T and their neighborhoods are smaller than any interior minima. Consequently, it is clear that the end-point values that bound the region of C_T also must be determined. That is, the authors are interested in defining the least and the largest possible values of C_T for a given set of boundary conditions (given v_{c1}, r_0, σ). These given boundary conditions define the end points for all possible orbits (regardless of conic space figure). First, consider the least possible value, with reference to applicable geometric conditions of the orbital hodograph.

For given values of v_{c1} and $v_{c2} < v_{c1}$ (i.e., $r_0 < 1$), the transfer angle σ will be varied from π to 0 to obtain the complete region of possible C_T^2. As shown in Fig. 3a, only one value of C_T^2 exists for $\sigma = \pi$; that is,

$$C_T^2 = (v_{c1}^2 + v_{c2}^2)/2 \tag{4}$$

or

$$x^2 = (1 + r_0^2)/2 \tag{5}$$

In this case, $\phi_{T1} = 0$ so that $\phi_{T2} = \sigma = \pi$.

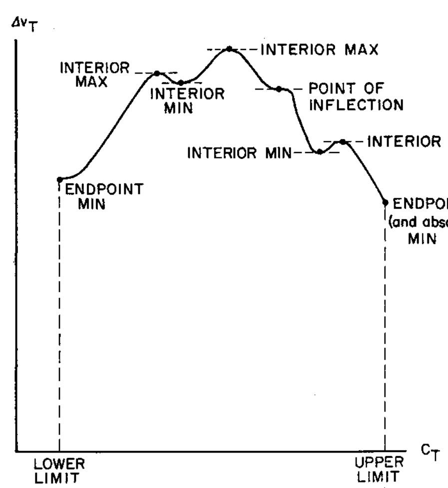

Fig. 2 Hypothetical function with end point absolute minimum

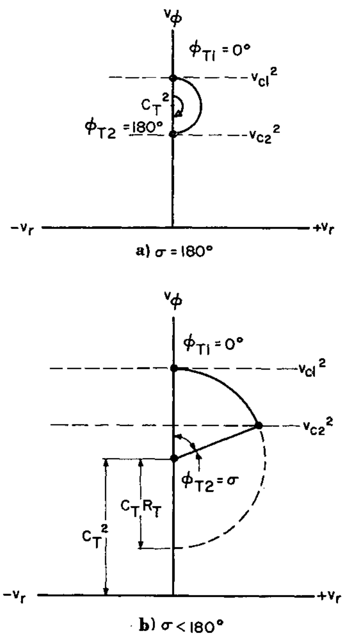

Fig. 3 Hodograph conditions for minimum $C_T(v_{c1} > v_{c2}; R_T < C_T)$

As the required transfer angle decreases from $\sigma = \pi$, the *least* possible value of C_T for each required σ value is obtained by selecting $\phi_{T1} = 0$, if possible, as shown in Fig. 3b. Upon study of the transfer hodograph geometry, it is clear that any value of ϕ_{T1} other than zero will result in a value of C_T greater than this least value. By use of Eq. (1l),

$$(v_{c1}^2 - C_T^2)\cos\sigma = v_{c2}^2 - C_T^2 \tag{6}$$

since

$$\cos\phi_{T1} = 1$$

$$\cos\phi_{T2} = \cos\sigma$$

The resulting solution for C_T^2 is then

$$C_T^2 = \left[\frac{(v_{c2}^2/v_{c1}^2) - \cos\sigma}{1 - \cos\sigma}\right] v_{c1}^2 \tag{7}$$

or

$$x^2 = \frac{r_0^2 - \cos\sigma}{1 - \cos\sigma} \tag{8}$$

Equation (7) provides the least possible value for C_T^2 until the value of σ is obtained where $C_T^2 = C_T R_T$, as shown in Fig. 4a, when

$$x^2 = \frac{r_0^2 - \cos\sigma}{1 - \cos\sigma} = \frac{1}{2} \tag{9}$$

or

$$\sigma = \operatorname{arc}\cos(2r_0^2 - 1) \tag{10}$$

A hodograph circle tangent to the origin represents a parabolic orbit. If the circle center continues to be lowered with the same given value of R_T, so that the hodograph circle crosses the v_r axis, then hyperbolic orbits are defined. Consequently, the least possible value of C_T^2 for an elliptical transfer orbit when $\sigma < \operatorname{arc}\cos(2r_0^2 - 1)$ must be greater than $C_T^2 = v_{c1}^2/2$. In this case (see Fig. 4b) $\phi_{T1} \neq 0$, so that

$$C_T R_T = C_T^2 = (v_{c2}^2 - C_T^2)/\cos\phi_{T2} \tag{11}$$

and consequently

$$C_T^2 = v_{c2}^2/(1 + \cos\phi_{T2}) \tag{12}$$

Since

$$\phi_{T2} = \phi_{T1} + \sigma$$

and

$$\phi_{T1} = \operatorname{arc}\cos[(v_{c1}^2 - C_T^2)/C_T R_T)] \tag{13}$$

in this case, the least value of C_T^2 (when $r_0 < 1$) is

$$C_T^2 = \frac{(v_{c1}^2 + v_{c2}^2) - v_{c1}v_{c2}[2(1 + \cos\sigma)]^{1/2}}{1 - \cos\sigma} \tag{14}$$

or

$$x^2 = \frac{(1 + r_0^2) - r_0[2(1 + \cos\sigma)]^{1/2}}{1 - \cos\sigma} \tag{15}$$

The largest possible value of C_T^2, for given v_{c1}, v_{c2}, and σ is obtained in a similar fashion. However, all transfer orbits obtained with the largest possible value of C_T^2 will be elliptical, because such transfer hodographs will never intersect the v_r axis. Consequently, in this case, the analytical expression will be continuous for all σ.

All analytic definitions for the least and the largest possible values of C_T^2, for any value of r_0, are presented in Tables 3 and 4. Note that the restriction of the least possible value of C_T to elliptical transfer orbits is for convenience only. Further study of the variation of Δv_T as a function of C_T^2 shows that hyperbolic transfer orbits will require a greater total velocity

Table 3 C_T^2 end points ($v_{c2} < v_{c1}$ or $r_0 < 1$)

Least value of C_T^2:

$$C_T^2 = \frac{v_{c2}^2 - v_{c1}^2\cos\sigma}{1 - \cos\sigma} \tag{3A}$$

or

$$x^2 = \frac{r_0^2 - \cos\sigma}{1 - \cos\sigma} \tag{3B}$$

for $\pi \geqslant \sigma \geqslant \operatorname{arc}\cos(2r_0^2 - 1)$ or $r_0^2 \geqslant \frac{1}{2}(1 + \cos\sigma)$

or

$$C_T^2 = \frac{(v_{c1}^2 + v_{c2}^2) - v_{c1}v_{c2}[2(1 + \cos\sigma)]^{1/2}}{1 - \cos\sigma} \tag{3C}$$

or

$$x^2 = \frac{(1 + r_0^2) - r_0[2(1 + \cos\sigma)]^{1/2}}{1 - \cos\sigma} \tag{3D}$$

for $\operatorname{arc}\cos(2r_0^2 - 1) > \sigma > 0$ or $r_0^2 < \frac{1}{2}(1 + \cos\sigma)$

Largest value of C_T^2:

$$C_T^2 = \frac{v_{c1}^2 - v_{c2}^2\cos\sigma}{1 - \cos\sigma} \tag{3E}$$

or

$$x^2 = \frac{1 - r_0^2\cos\sigma}{1 - \cos\sigma} \tag{3F}$$

for $\pi \geqslant \sigma > 0$

Table 4 $C_T{}^2$ end points ($v_{c2} > v_{c1}$ or $r_0 > 1$)

Least value of $C_T{}^2$:

$$C_T{}^2 = \frac{v_{c1}{}^2 - v_{c2}{}^2 \cos\sigma}{1 - \cos\sigma} \quad \text{for} \quad \pi \geqslant \sigma \geqslant \arccos\left(\frac{2 - r_0{}^2}{r_0{}^2}\right) \tag{4A}$$

or

$$x^2 = \frac{1 - r_0{}^2 \cos\sigma}{1 - \cos\sigma} \quad \text{or} \quad r_0{}^2 \leq \frac{2}{1 + \cos\sigma} \tag{4B}$$

$$C_T{}^2 = \frac{(v_{c1}{}^2 + v_{c2}{}^2) - v_{c1}v_{c2}[2(1 + \cos\sigma)]^{1/2}}{1 - \cos\sigma} \quad \text{for} \quad \arccos\left(\frac{2 - r_0{}^2}{r_0{}^2}\right) > \sigma > 0 \tag{4C}$$

or

$$x^2 = \frac{(1 + r_0{}^2) - r_0[2(1 + \cos\sigma)]^{1/2}}{1 - \cos\sigma} \quad \text{or} \quad r_0{}^2 > \frac{2}{1 + \cos\sigma} \tag{4D}$$

Largest value of $C_T{}^2$:

$$C_T{}^2 = \frac{v_{c2}{}^2 - v_{c1}{}^2 \cos\sigma}{1 - \cos\sigma} \tag{4E}$$

or

$$x^2 = \frac{r_0{}^2 - \cos\sigma}{1 - \cos\sigma} \tag{4F}$$

for $\pi \geqslant \sigma > 0$

increment than the elliptical transfer orbits. Also, since Δv_T is a continuous function of C_T, the interior minimum will be established as the absolute minimum if it is shown to be smaller than at the end points.

IV. General Solution for Nonaligned Elliptical Orbits

Equations (1B–1D) are available for minimization of the total velocity increments for nonaligned elliptical orbits. The total velocity increment must be minimized with respect to the hodograph parameter C_T. When the required optimum value of C_T is obtained, the two remaining dependent parameters (R_T, Ψ_T) are defined by Eqs. (1G–1I). Also, the various velocity vector relations at each transfer point can be found by use of the analytic relations of Table 2 and Ref. 13.

Those values of C_T which yield interior minimum and maximum values of Δv_T are obtained as the positive real roots of Eq. (3). Use of Eqs. (1B–1D) in Eq. (3) results, upon reduction, in the eighth-order polynomial (or octic) Eq. (5A) shown in Table 5. (Note that the terms c_1 and c_2 are defined in Table 1.) The constant coefficients K_i ($i = 0, 1, 2, \ldots 8$) are defined by the boundary conditions of the problem, i.e., the initial transfer point 1 on the initial orbit A and the final transfer point 2 on the final orbit B. The one positive real root of the polynomial equation, which results in the minimum total velocity increment Δv_T, provides the relative minimum within the interior of the realizable region for C_T. If this value of Δv_T is smaller than the Δv_T solutions for the end-point values of C_T, then that root is the absolute minimum velocity increment solution.

Machine computation for this absolute minimum will provide direct solution by use of Eqs. (1B–1D, 6, 3A–3F, 4A–4F, and 5A) for the specific trajectory end-point conditions. In addition, it would be desirable to determine those conditions (in analytic form) for which the interior minimum provides the absolute minimum solution. Further investigation by machine computation certainly could answer this question, although further analysis might provide such conditions in explicit form without machine computation.

V. Special Solution for Circular Orbits

In order to demonstrate the usefulness of this optimization study, the polynomial equation for the general solution [Eq. (5A)] was reduced to that of Eq. (6A) (Table 6) for the special solution defining optimum transfer between circular orbits. If both the initial and final orbits are circular, then $c_1 = c_2 = 0$. Also, all constant coefficients K_i ($i = 0, 1, 2, \ldots$ 8) simplify considerably, $K_6 = 0$, and all coefficients can be reduced further by the factor $4 \sin^2\sigma$. Note that all constant coefficients K_i (or K_i) are functions of the transfer angle σ and the circular orbit velocity ratio r_0.

Machine solutions were obtained for Eq. (6A) and corresponding end points for $r_0 < 1$. It was determined that the absolute minimum always is provided by an interior point resulting from solution of the polynomial equation. For example, Figs. 5 and 6 provide minimum total velocity increment solutions of the transfer orbit for $\sigma = 50°$. In Fig. 5, the hodograph characteristics x, y, and e_T for the transfer orbits that require the least total velocity increment are presented as functions of the velocity ratio r_0. Also, the least permissible value x_{LL} and the greatest permissible value x_{UL} of the possible range of the variable x for the given end-point conditions of r_0 and σ are presented as dotted lines. Then the region lying between these curves contains all permissible values for solution; obviously, the curve for optimum x lies entirely within this region.

In Fig. 6, the velocity characteristis z, u, w, and $u + w$ are presented as functions of r_0 for the hodograph solutions of x shown in Fig. 5. The total velocity increment $(u + w)_{LL}$ re-

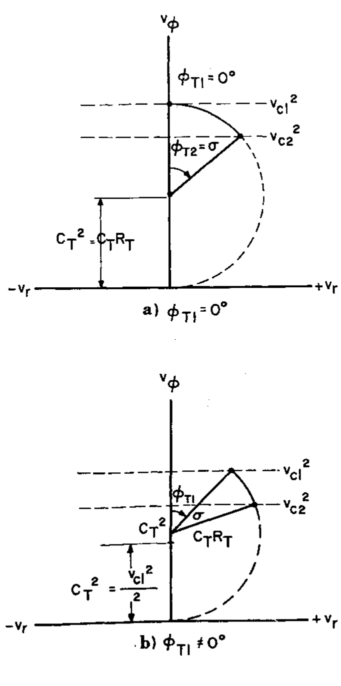

Fig. 4 Hodograph conditions for minimum $C_T(v_{c1} > v_{c2}$; $R_T = C_T)$

Table 5 The eighth-order polynomial for general solution

$$K_8x^8 + K_7x^7 + K_6x^6 + K_5x^5 + K_4x^4 + K_3x^3 + K_2x^2 + K_1x + K_0 = 0 \tag{5A}$$

where

$$K_8 = A[(G^2 - B^2) + 4A(D - H)] \tag{5B}$$
$$K_7 = 4A[2A(J - E) + (DG - BH)] + [BG(G - B)] \tag{5C}$$
$$K_6 = 2A[4(BJ - EG) + (GJ - BE)] + [DG^2 - B^2H] \tag{5D}$$
$$K_5 = 4A[2F(G - B) + (DJ - EH)] + [2BG(J - E) + B^2J - EG^2] \tag{5E}$$
$$K_4 = A[(J^2 - E^2) - 8F(D - H)] + [F(G^2 - B^2)] + [2(DGJ - BEH)] \tag{5F}$$
$$K_3 = -4F[2A(J - E) + (DG - BH)] - [2EJ(G - B)] + [BJ^2 - E^2G] \tag{5G}$$
$$K_2 = 2F[-4(BJ - EG) + (GJ - BE)] + [DJ^2 - E^2H] \tag{5H}$$
$$K_1 = -4F[2F(G - B) + (DJ - EH)] - [EJ(J - E)] \tag{5I}$$
$$K_0 = F[(J^2 - E^2) + 4F(D - H)] \tag{5J}$$

and

$$A = (1 - \cos\sigma)^2 \tag{5K}$$
$$B = (2c_1/C_A)(1 - \cos\sigma) \tag{5L}$$
$$D = (1/C_A^2)[v_{c1}^4 \sin^2\sigma + 2C_A^2(1 - \cos\sigma)(v_{c1}^2 \cos\sigma - v_{c2}^2) + c_1^2] \tag{5M}$$
$$E = (2/C_A)[v_{c1}^4 \sin^2\sigma - c_1(v_{c1}^2 \cos\sigma - v_{c2}^2)] \tag{5N}$$
$$F = v_{c1}^4 - 2v_{c1}^2v_{c2}^2 \cos\sigma + v_{c2}^4 \tag{5O}$$
$$G = (2c_2/C_B)(\cos\sigma - 1) \tag{5P}$$
$$H = (1/C_B^2)[v_{c2}^4 \sin^2\sigma + 2C_B^2(\cos\sigma - 1)(v_{c1}^2 - v_{c2}^2 \cos\sigma) + c_2^2] \tag{5Q}$$
$$J = (2/C_B)[v_{c2}^4 \sin^2\sigma - c_2(v_{c1}^2 - v_{c2}^2 \cos\sigma)] \tag{5R}$$

sulting from x_{LL} and the total velocity increment $(u + w)_{UL}$ resulting from x_{UL} are presented as dotted lines. Note that the curve for optimum $u + w$ lies below both dotted lines. All permissible values of x in Fig. 5 provide $u + w$ values that lie above this optimum curve. Consequently, the absolute minimum total velocity increment is provided by the interior point resulting from the solution of the polynomial equation.

Solutions for x resulting in minimum velocity increment transfer have been obtained by digital computation for 1) $\sigma = 10°$ to $170°$ in $10°$ steps and $\sigma = 170°$ to $178°$ in $2°$ steps; and 2) $r_0 = 0.1$ to 0.9 in 0.1 steps and $r_0 = 0.9$ to 0.98 in 0.02 steps. In addition to these solutions, all transfer orbit and transfer point characteristics were determined and presented in graphical form.[14] These data can be presented graphically as a function of the transfer angle σ with the circular orbit velocity ratio r_0 as parameter, or in converse form. The optimum transfer orbit variable, the total velocity increment, and the transfer time are presented in Figs. 7–9, respectively, as a function of σ, for r_0 varied from 0.1 to 0.9. Study of these data results in the following observations:

1) For selectable initial and final trajectory end points, the required velocity increment $(u + w)$ is minimum when $\sigma = 180°$ (i.e., Hohmann transfer).

2) For selectable initial and final trajectory end points, the required velocity increment $(u + w)$ for σ values about 180° is not much greater than the Hohmann transfer increment (e.g., an increase of less than 10% for $\sigma = 160°$).

3) For given σ, the initial velocity increment (u) is a monotonically increasing function of decreasing values of r_0, whereas the final velocity increment (w) is not.

4) For given r_0, the transfer time is minimum for r_0 less than about 0.3 and for σ about 100° to 120°.

Table 6 The eighth-order polynomial equation for circular orbit solution

$$K_8x^8 + K_7x^7 + K_6x^6 + K_5x^5 + K_4x^4 + K_3x^3 + K_2x^2 + K_1x + K_0 = 0 \tag{6A}$$

where

$$K_8 = 4A^2(D - H) = 12 \sin^2\sigma(1 - \cos\sigma)^4(1 - r_0^2) \tag{6B}$$
$$K_7 = 8A^2(J - E) = 16 \sin^2\sigma(1 - \cos\sigma)^4(r_0^3 - 1) \tag{6C}$$
$$K_6 = 0 \tag{6D}$$
$$K_5 = 4A(DJ - EH) \tag{6E}$$
$$= 8 \sin^2\sigma(1 - \cos\sigma)^2(1 - r_0)[2(1 - \cos\sigma)(1 + r_0^2)(1 + r_0 + r_0^2) - 3r_0^2 \sin^2\sigma]$$
$$K_4 = A[(J^2 - E^2) - 8F(D - H)] \tag{6F}$$
$$= 4 \sin^2\sigma(1 - \cos\sigma)^2[6(r_0^2 - 1)(1 - 2r_0^2 \cos\sigma + r_0^4) + (r_0^6 - 1)\sin^2\sigma]$$
$$K_3 = 8AF(E - J) \tag{6G}$$
$$= 16 \sin^2\sigma(1 - \cos\sigma)^2(1 - r_0^3)(1 - 2r_0^2 \cos\sigma + r_0^4)$$
$$K_2 = DJ^2 - E^2H \tag{6H}$$
$$= 4 \sin^4\sigma(1 - \cos\sigma)(r_0^4 - 1)[r_0^2(1 + 3 \cos\sigma) - 2(r_0^4 + 1)]$$
$$K_1 = 4F(EH - DJ) - EJ(J - E) \tag{6I}$$
$$= 8 \sin^2\sigma(1 - r_0)(1 - 2r_0^2 \cos\sigma + r_0^4)[3r_0^2 \sin^2\sigma - 2(1 + r_0 + r_0^2)(1 + r_0^2)(1 - \cos\sigma)] + 8r^3 \sin^6\sigma(1 - r_0^3)$$
$$K_0 = F[(J^2 - E^2) + 4F(D - H)] \tag{6J}$$
$$= 4 \sin^2\sigma(1 - 2r_0^2 \cos\sigma + r_0^4)[(r_0^6 - 1) \sin^2\sigma + 3(1 - r_0^2)(1 - 2r_0^2 \cos\sigma + r_0^4)]$$

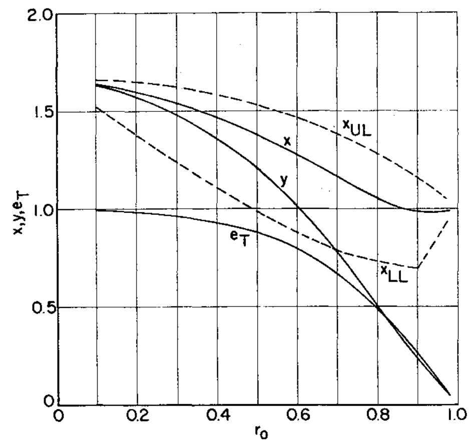

Fig. 5 Hodograph characteristics for optimum transfer between circular orbits ($\sigma = 50°$)

The solutions are presented for $0 < \sigma \leq \pi$. Since Eq. (6A) is a function of $\sin^{2n}\sigma$ and $\cos^n\sigma$, it is identical for σ and $(2\pi - \sigma)$, that is,

$$\sin^{2n}(2\pi - \sigma) = \sin^{2n}\sigma$$

$$\cos^{n}(2\pi - \sigma) = \cos^{n}\sigma$$

where $n = 1, 2, 3$, or 4. Consequently, the solutions for x are also valid for $(2\pi - \sigma)$. Also, Eqs. (2B) and (2C) for u and w, respectively, provide identical solutions of u and w for σ and $(2\pi - \sigma)$.

The solutions provided in this report have been obtained for $0 < r_0 < 1.0$. Solutions for the hodograph parameter C_T referred to the final circular orbit velocity v_{c2} can be obtained directly for $r_0 > 1.0$ from the solution x for $0 < r_0 < 1.0$ (Fig. 7). It has been determined by analysis that the numerical solution for $x' = C_T/v_{c2}$ is obtained directly from the solution for x vs r_0 by selecting the abscissa value as the reciprocal of the given velocity ratio $r_0 > 1.0$. For example, suppose the solution x' is required when $r_0 = 2$ and $\sigma = 50°$. Referring to Fig. 5 or 7, the value of x' is obtained at abscissa value of 0.5; i.e., $x' = 1.38$.

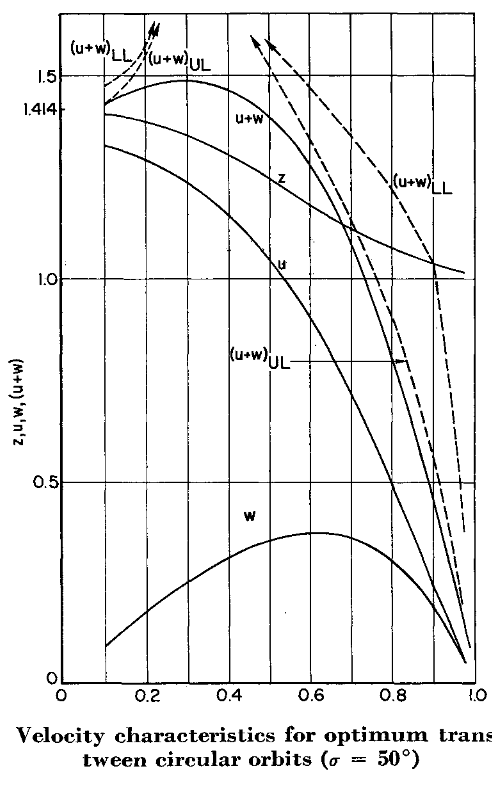

Fig. 6 Velocity characteristics for optimum transfer between circular orbits ($\sigma = 50°$)

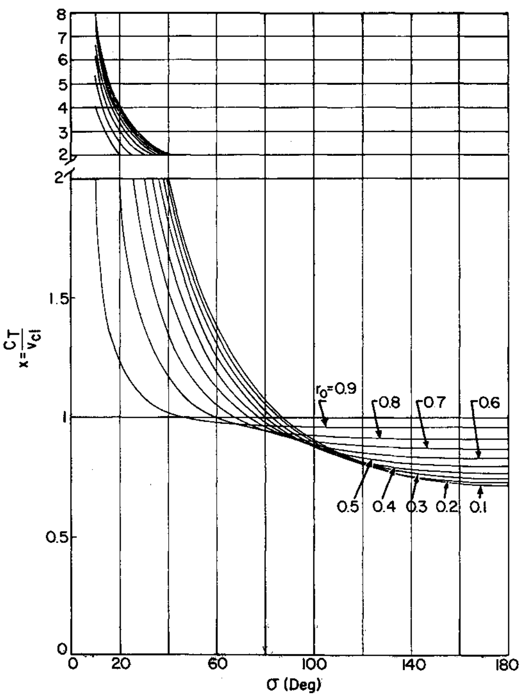

Fig. 7 Variation of hodograph parameter x

VI. Summary and Conclusions

The total velocity increment required for two-impulse coplanar orbital transfer is defined by an equation that is an algebraic function of the orbital hodograph parameter C_T and the trajectory end-point conditions. Consequently, the minimum total velocity increment solutions for specified trajectory end-point conditions are attainable directly by methods of the differential calculus. As a result, interior minima are provided by an eighth-order polynomial (or octic) equation with constant coefficients. Determination of the absolute minimum is obtained by comparison of the total velocity increments calculated by use of one of the positive real roots of the octic equation and the end points of the permissible range of C_T. Since all transfer characteristics are functions of the trajectory end-point conditions and the value of C_T giving this absolute minimum, complete data on the minimum total velocity increment problem are available.

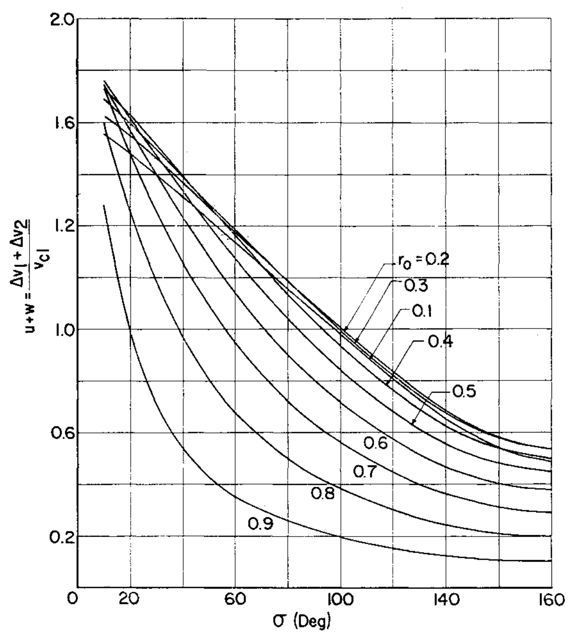

Fig. 8 Variation of total velocity increment

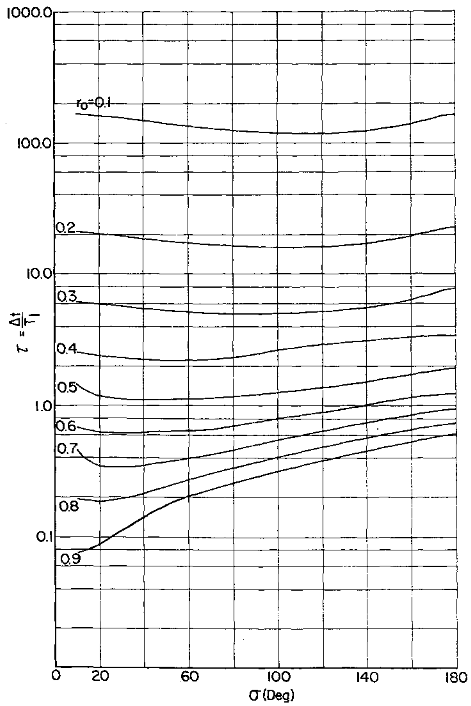

Fig. 9 Variation of transfer time

Note that the solutions are directly and explicitly obtainable for arbitrarily specified trajectory end points.

The octic Eq. (5A) enables solutions to be obtained for the general problem of transfer between end points lying on coplanar nonaligned elliptical orbits. Reduced forms of this octic equation are available directly for special classes of trajectory end points, e.g., transfer between circular orbits or between a circular and an elliptical orbit. The reduced octic Eq. (6A) has been found to provide the direct solution for transfer between circular orbits, since the absolute minima are always interior to the permissible range of C_T for this problem. The normalized optimum transfer orbit variable x, the initial, final, and total velocity increments u, w, $u + w$, and the transfer time τ are presented in Figs. 7–9 as a function of the transfer angle σ(for $0 < \sigma \leq 180°$), with the circular orbit velocity ratio r_0 (for $0 < r_0 < 1.0$) as parameter. The complete optimum transfer orbit and transfer point characteristics were determined and presented in Ref. 14. Moreover, it has been shown that the solutions for x, u, and w obtained for $0 < \sigma \leq \pi$ also are valid for $\pi \leq \sigma < 2\pi$ by mirror symmetry about $\sigma = \pi$. Also, solutions for C_T (normalized with respect to the initial circular orbit velocity v_{c1}) obtained for $0 < r_0 < 1.0$ provide solutions for C_T (normalized with respect to the final circular orbit velocity v_{c2}) valid for $r_0 > 1.0$.

Study of the solutions for transfer between circular orbits confirms previous work[5] that has shown that the minimum total velocity increment (for two-impulse transfer with selectable end points) is obtained at $\sigma = \pi$. However, it also is clear that transfer by use of an anomaly difference σ up to $\pm 20°$ from Hohmann transfer ($\sigma = \pi$) results in less than 10% increase in the required total velocity increment. Consequently, trajectory end-point conditions for transfer may deviate appreciably from the Hohmann transfer conditions without undue or intolerable penalties upon propulsion and vehicle staging design.

Further study, by analysis and machine computation, should provide extensive knowledge of minimum total velocity increment solutions for the general problem of trajectory end points lying on elliptical orbits. In any case, the availability of algebraic equations, which provide explicit solutions directly, enables serious consideration of on-board guidance systems with present state-of-the-art system and component design techniques. It is suggested that versatile man-machine systems for transfer guidance can be conceived upon development of control and command display systems and computation systems that mechanize the equations of the orbital hodograph and the optimization process.

References

[1] Sutton, G. P., *Rocket Propulsion Elements* (John Wiley and Sons Inc., New York, 1949), 2nd ed. 1958, p. 280.

[2] Lawden, D. F., "Inter-orbital transfer of a rocket," J. Brit. Interplanet. Soc. **11**, 321 (1952).

[3] Van Gelder, A., Beltrani, E., and Munick, H., "On minimum time minimum fuel rendezvous," Soc. Indust. Appl. Math. J. **9**, 474–480 (1961).

[4] Ting, L., "Optimum orbital transfer by several impulses," Astronaut. Acta **VII** (1961).

[5] Hohmann, W., *Die Erreichbarkeit der Himmelskörper* (Oldenbourg, Munich, 1925).

[6] Haynes, G. W., "Minimum velocity transfer between coplanar circular orbits, employing an arbitrary number of impulses," Martin-Denver Research Rept. R-60-47 (September 1960).

[7] Hoelker, R. F. and Silber, R., "The bi-elliptical transfer between circular copolanar orbits," *Proceedings of the 4th AFBMD/STL Symposium 1959* (Pergamon Press, New York, 1961), Vol. III, pp. 164–175.

[8] Silber, R., "Survey of characteristic velocity requirements for two-impulse transfers between circular and coplanar exterior elliptical orbit with exposition of local and overall optimum solutions," NASA Rept. MTP-AERO-60-10 (December 9, 1960).

[9] Vargo, L. G., "Optimal transfer between coplanar terminals in a gravitational field," *Advances in Astronautical Sciences* (Plenum Press, New York, 1958), Vol. 3, pp. 20-1 through 20-9.

[10] Munick, H., Kelley, H. J., Taylor, G. E., and McGill, R., "Analytic solution for optimum ellipse-to-circle transfer," Grumman Aircraft Engineering Corp. Research Rept. RE-133 (May 1960).

[11] Altman, S. P., "Orbital transfer for satellites," *Proceedings of the 4th AFBMD/STL Symposium 1959* (Pergamon Press, New York, 1961), Vol. III, pp. 132–144.

[12] Altman, S. P. and Pistiner, J. S., "Hodograph analysis of the orbital transfer problem for coplanar nonaligned elliptical orbits," ARS J. **31**, 1217–1225 (1961).

[13] Altman, S. P. and Pistiner, J. S., "Velocity vector direction changes for coplanar orbital transfer," *Advances in Astronautical Sciences* (Plenum Press, New York, 1962), Vol. 9; also Am. Astronaut. Soc. Preprint 61-75 (1961).

[14] Altman, S. P. and Pistiner, J. S., "Minimum velocity increment solution for two-impulse coplanar orbital transfer," United Aircraft SCTM 124 (January 1962).